Factoring a Catalan Number into Chebyshev's Segments

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Abstract. This paper describes the "cutting" of the Catalan number decomposition in Chebyshev's segments. The segment content is not calculated and is selected from a set of primes according to the segment bounds. In order to decompose the nth Catalan number it is enough to calculate primes up to $\sqrt{2n}$, i.e. to get a tiny core that corresponds to the factoring a so-called slight Catalan number.

Key Words: Prime factorization, factoring, decomposition, Catalan numbers, Chebyshev, Cayley.

1. Introduction

A Prime Factorization of large numbers takes time to compute. Often factoring special numbers such as Fermat numbers, Mersenne numbers, Catalan numbers, Motzkin numbers, etc. are used to validate methodologies and algorithms for decomposition of huge integers. This paper is related to Catalan numbers that occur in numerous combinatorial applications (see, e.g., [Stan15]).

Cayley formula. Special numbers are characterized by mutual relationships between elements of the corresponding sequences, and this simplifies a Prime Factorization. Relationships are implemented by recurrence and analytical formulas.

Let $\mathbb{N}$ denote non-negative integers. The nth Catalan number is defined so

$$C(n) = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}, \quad n \in \mathbb{N}. \tag{1}$$

The first Catalan numbers for $n = 0, 1, 2, 3, \ldots$ are

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, … (see [A000108]).

We are interested in a composite Catalan number $C(n) > 5$, i.e. $n > 3$. So then everywhere index $n \geq 4$. In (1) we can reduce a number of operands. Let's separate even factors from odd factors in the numerator

$$(2n)! = 2 \cdot 4 \cdot 6 \cdots 2n \times 1 \cdot 3 \cdot 5 \cdots (2n-1) = 2^n \times n! \times (2n-1)!!$$

After the cuts it will receive the dependency (see [Wei16])

$$C(n) = 2^n \times (2n-1)!! \big/ (n+1)! \tag{2}$$

A similar expression was published by Arthur Cayley in 1859, counting a number of binary trees (see historical review [Pak14)]). So let us call the equation (2) *Cayley formula*. The combination of an ordinary factorial and an odd double factorial simplifies a Prime factorization of Catalan numbers.

Let's repeat the procedure with the ordinary factorial

$$(n+1)! = 2\cdot 4\cdot 6\cdots 2\left\lceil\frac{n}{2}\right\rceil \times 1\cdot 3\cdot 5\cdots(2\left\lfloor\frac{n}{2}\right\rfloor+1) = 2^{\lceil n/2\rceil} \times \left\lceil\frac{n}{2}\right\rceil! \times (2\left\lfloor\frac{n}{2}\right\rfloor+1)!!$$

Obviously, $2\lceil ½n\rceil$ and $2\lfloor ½n\rfloor+1$ are the largest even integer and the largest odd integer, respectively, that do not exceed $n+1$. Since $\lceil ½n\rceil+\lfloor ½n\rfloor = n$, then

$$C(n) = 2^{\lfloor ½n\rfloor} \times (2\lfloor ½n\rfloor+3)\cdot(2\lfloor ½n\rfloor+5)\cdots(2n-1) \Big/ \lceil ½n\rceil! \qquad (3)$$

The expression (3) is convenient and practical. For example, in order to calculate a number of factors of 2 it is sufficient to decompose the factorial $\lceil ½n\rceil!$

In equations (1–3) all factors are less than $2n$. Let $\mathbb{P}$ denote the set of prime numbers. For the nth Catalan number, let $\mathrm{FB}(n) = {}_{\mathrm{p}}(1, 2n) \subset \mathbb{P}$, the *prime interval*, i. e. the open subset of prime numbers up to $2n$, and let us call this interval the *Factor Base* of $C(n)$. Let F_n denote the multiset of all prime factors of $C(n)$. In general, $F_n \not\subset \mathrm{FB}(n)$ due to prime powers (multiple primes).

Chebyshev's segments. Composition of primes in the multiset F_n is mainly determined by the binomial coefficient $\binom{2n}{n}$. In the middle of the 19th century Chebyshev drew attention to the interval ${}_{\mathrm{p}}(n, 2n) \subset \mathbb{P}$ [Pom15]. This open set of primes is fully included in the Prime Factorization of the central binomial coefficient. Similarly, each prime number from the open interval ${}_{\mathrm{p}}(n+1, 2n) \subset \mathrm{FB}(n)$ is included into F_n. Let's call such intervals *Chebyshev's segments*. State the obvious but important theorem.

Theorem 1.1. *The Prime Factorization of the nth Catalan number contains exactly once each prime from the Chebyshev's segment* ${}_{\mathrm{p}}(n+1, 2n) \subset \mathbb{P}$.

This paper describes the family of noncrossing Chebyshev's segments in the decomposition of the nth Catalan number; the number of these segments less than $\sqrt{n/2}$. The interval ${}_{\mathrm{p}}(n+1, 2n)$ is most extensive, so let's call it the *main segment*. Other Chebyshev's segments are reduced as we move down the base $\mathrm{FB}(n)$.

Gaps between the Chebyshev's segments are closed subsets of $\mathbb{P}$, a sort of *no-go zone* in which there are no prime factors of the Catalan number. No-go zones are also reduced during the descent down to $\mathrm{FB}(n)$. The longest no-go zone is convenient to consider the infinite set of primes ${}_{\mathrm{p}}[2n, \infty)$. The bounds between Chebyshev's segments and no-go zones are defined using the Catalan number indexes.

We call the family of Chebyshev's segments a *train* of the Catalan number and the vast majority of primes are combined in this train. At the same time, the area of multiple primes or a *core* of the nth Catalan number is compressed to size of the prime interval ${}_{\mathrm{p}}(1, \sqrt{2n})$. Thus almost all base $\mathrm{FB}(n)$ can be divided into Chebyshev's segments.

Example 1.1. To decompose the 1000th Catalan number it is sufficient to calculate only 14 primes from the interval $_{\text{p}}(1, \sqrt{2000})$. Other prime factors are selected from Chebyshev's segments. For example, the main segment $_{\text{p}}(1001, 2000)$ "furnishes" 135 primes into the multiset F_{1000}. From the adjacent Chebyshev's segment $_{\text{p}}(1001/2, 2000/3)$ transferred 26 primes. In the no-go zone $_{\text{p}}[2000/3, 1001]$ there is no prime numbers that divide $C(1000)$, i.e. $F_{1000} \cap {}_{\text{p}}[2000/3, 1001] = \emptyset$ and $_{\text{p}}(1001, 2000) \cup {}_{\text{p}}(1001/2, 2000/3) \subset F_{1000}$. In the multiset F_{1000} there are 214 primes and prime powers. The Chebyshev's segments cover 197 primes, and only 17 prime factors fall into the core of $C(1000)$.

Interest is a so-called *slight Catalan number*, the prime factorization of which is a tiny core – area of prime powers of the corresponding "heavyweight". At the end of this paper will talk a little bit about slight Catalan numbers.

2. Factor base of Catalan Numbers

In general, the multiset F_n has primes and prime powers. All primes from the main segment $_{\text{p}}(n+1, 2n) \subset \text{FB}(n)$ are not repeated into F_n. Obviously, the numeric space of $\text{FB}(n)$ can be divided into two disjoint areas. The *upper base* $\text{UB}(n)$ contains distinct primes that cannot theoretically be repeated into F_n. The prime powers and some primes are selected from the *root base* $\text{RB}(n)$. Let $\text{ppb}(n)$ denote the *prime power border*. It could be the bottom of the main segment if this segment is the only one. Below we will show that it is not, and in general, $\text{ppb}(n) \ll n+1$.

Everyone can find a similar border in the Prime Factorization of any natural number. In order to factor a positive integer m it is sufficient to verify primes up to $\sqrt{m}$. Only in this range a prime can be repeated, i.e. squares are possible.

No-go zones. Consider in the upper base $\text{UB}(n)$ those primes that are adjacent to the bottom of main segment $_{\text{p}}(n+1, 2n)$. Let's write in detail the factorials in Cayley formula (2):

$$C(n) = 2^n \times \frac{1\cdot 3\cdot 5\cdot 7\cdots (2n-3)(2n-1)}{1\cdot 2\cdot 3\cdot 4\cdots n(n+1)}. \qquad (4)$$

In the denominator the last factor $n+1$ may be a prime in the case of even n. But this factor is in the numerator, and also in a single copy. So $n+1 \notin F_n$. The odd factor n is also available in a single copy in the both factorials, so $n \notin F_n$. The situation persists if we continue to descend on natural numbers until we get to $⅔n$. This number may be an integer, and therefore even and composite. So, $⅔n \notin F_n$. Thus any prime $⅔n < q \le n+1$ is contained in the single copy into the both factorials, namely: (a) in the numerator the second copy $3q > 2n$; (b) in the denominator we have $2q > 2\times⅔n = n + ⅓n > n+1$ (let's remember $n \ge 4$).

As a result, the first (probably not the last) no-go zone is defined in the upper base, i.e. $F_n \cap {}_{\text{p}}[⅔n, n+1] = \emptyset$. Additionally, we have lowered the border $\text{ppb}(n) < ⅔n$.

Let's do a final check of any prime $\frac{1}{2}(n+1) < p < \frac{2}{3}n$. In the numerator we get the second copy of thanks to an odd composite number $3p < 3\times\frac{2}{3}n = 2n$. In the denominator p remains in a single copy $(2p > n+1)$, so $p \in F_n$. Clearly, these primes form the next Chebyshev's segment. Let foregoing be regarded as a proof of the following theorem.

Theorem 2.1. *Let prime* $p \in {}_{\mathrm{p}}[\frac{2}{3}n, n+1] \subset \mathbb{P}$. *Then* $p \nmid C(n)$.

Two segments and the no-go zone between them are reduced fourfold the space of the factor base for a selection of the remaining primes.

Power border. Moving down the factor base, starting from the main segment and passing through the no-go zone, we found the next Chebyshev's segment. Obviously, there is another no-go zone lower. So we will be a long time to reach the power border. The following example will help accelerate this process.

Example 2.1. Let's choose the prime 101 and let's try to find the smallest index n such that $101^2 \mid C(n)$. In this case, $\mathrm{ppb}(n) = 101$ or very close. This experiment will help to further to define the power border for an arbitrary Catalan number. In (4) let's write some composite factors of the numerator and of the denominator that are divisible by 101:

numerator 1, 3, 5, 7, … , 101, … , 101×3, … , 101×5, … , 101×101, …
denominator 1, 2, 3, 4, … , 101, … , 101×2, … , 101×3, … , 101×101, …

In the case of $n \le 50$, the both factorials don't contain 101. So $101 \nmid C(n)$, $n \le 50$. In the following range $51 \le n < 100$ the factor 101 is only in the numerator in a single copy. Thus, we found the zone of favorable indexes for 101, i.e. $101 \mid C(n)$, $51 \le n < 100$. For $n = 100$ the symmetric factor 101 appears in the denominator, therefore, $101 \nmid C(100)$. This situation will persist in the range $100 \le n < 152$. This is the no-go zone indexes for 101.

Gradually increasing an index n, we get a series of favorable zones and no-go zones. In each favorable zone $101^2 \nmid C(n)$, and when we get to $2n - 1 = 101\times101$, then we obtain two factors 101 in the numerator at once. Thus, in the case $n = (101^2+1)/2 = 5101$ the prime power 101^2 divides $C(5101)$. So, we got a required Catalan number.

In Example 2.1 we received the smallest 5101th Catalan number that is divisible by 101^2. For the next prime 103 we get another index $n = (103^2+1)/2 = 5305$. So, $p^2 \nmid C(5101)$, $p > 101$, and therefore you can take as a power border, for example, the next even integer, i.e. $\mathrm{ppb}(5101) = 102$. We have the right to formulate the following theorem.

Theorem 2.2. *Let prime* $p > \sqrt{2n}$, then $p^k \nmid C(n)$, $k \ge 2$.

It is logical and convenient to take $\mathrm{ppb}(n) = \sqrt{2n}$. An integer $\mathrm{ppb}(n)$ is always an even number (in this case, n is an even number also), so $\mathrm{ppb}(n) \notin \mathrm{FB}(n)$. Let's give a more general theorem.

Theorem 2.3. *Let prime* $p > \sqrt[m]{2n}$, *then* $p^k \nmid C(n)$, $k \ge m$.

Remark 2.1. When we say, for example, about an even number $\frac{2}{3}n$ or a natural number $(n+1)/2$, we mean certain (or probable) values for n, namely: in the first case, n is divisible by 3, the second n is an odd number.

3. Train of Catalan numbers

For the nth Catalan number the border $\mathrm{ppb}(n) = \sqrt{2n}$ divides the base $\mathrm{FB}(n)$ into two disjoint subsets. The upper base $\mathrm{UB}(n) = {}_{\mathrm{p}}(\mathrm{ppb}(n), 2n)$ is a group of Chebyshev's segments S_i, that are separated by no-go zones Z_i. Chebyshev's segments accumulate the most primes that form a train of the Catalan number. For example, the train of the 1,000,000th Catalan number contains 99.7% of all prime factors.

The main segment is the longest, adjacent to the top of $\mathrm{FB}(n)$, is evident in analytical formulas and therefore is defined in the first turn. Let's assign this segment the first number, i.e. $S_1 = {}_{\mathrm{p}}(n+1, 2n)$. The bottom S_1 is adjacent the most extensive no-go zone, this zone will give the first number also, i.e. $Z_1 = {}_{\mathrm{p}}[\frac{2}{3}n, n+1]$. In addition, we discovered another Chebyshev's segment $S_2 = {}_{\mathrm{p}}(\frac{1}{2}(n+1), \frac{2}{3}n)$.

In this section we will consider the layout of segments and zones in the upper base $\mathrm{UB}(n)$. Let's try to estimate the total number of segments and zones.

Top of Chebyshev's segments. Previously we defined the upper open bound of the second segment is $\frac{2}{3}n$. Probable Prime number $\frac{2}{3}n-1 \in S_2$ (in this case, $3|n$). If we write the top open bound of segment S_1 as $2n/1$, we can notice certain regularity. Apparently, the upper open bound of S_3 is $2n/5$; for the next segment we will get $2n/7$, etc. Check it on the example.

Example 3.1. Let n =7500, then the top open bound of the 3rd segment should be 2×7500/(2×3–1) = 3000. Consider the twin primes 2999 and 3001. Let's refer to the formula (4) again.

In the odd double factorial 14999!! (the numerator of the fraction) the prime 2999 is in triplicate, namely: 2999, 3×2999, 5×2999. But in the ordinary factorial 7501! (the denominator of the fraction) there are only 2999 and 2×2999. So $2999 \in F_{7500}$. But for the next prime 3001 the composite number in the numerator 5×3001 is outside of FB(7500), so in the numerator and in the denominator there are exactly two copies of 3001. Therefore, $3001 \notin F_{7500}$.

The technique that used in Example 3.1 can be applied to check the top open bound of arbitrary Chebyshev's segment. Generalize the obtained result in the following theorem (the proof is in the next section).

Theorem 3.1. *In the train of the nth Catalan number the top open bound of kth Chebyshev's segment is* $2n/(2k-1)$, $k \geq 1$.

Bottom of Chebyshev's segments. Let us write the lower open bound of the main segment as $(n+1)/1$. Following the second Chebyshev's segment has a bottom bound $(n+1)/2$. Probably for the third segment we need to get $(n+1)/3$, etc. The trend is obvious and we can formulate a corresponding theorem (the proof, see the next section).

Theorem 3.2. *In the train of the nth Catalan number the bottom open bound of kth Chebyshev's segment is* $(n+1)/k,\ k \geq 1$.

So for the *n*th Catalan number, the train components take the following form:

$$S_k = {}_{\mathrm{p}}\big((n+1)/k,\ 2n/(2k–1)\big),\ \ Z_k = {}_{\mathrm{p}}[2n/(2k+1), (n+1)/k],\quad k \geq 1. \qquad (5)$$

How many segments in the train of the *n*th Catalan number? To estimate this value easy. The lower bounds of the *k*th Chebyshev's segments should be above the power border, i.e. $(n+1)/k > \mathrm{ppb}(n)$ or

$$k < (n+1)/\ \mathrm{ppb}(n) \approx n/\ \mathrm{ppb}(n) = \sqrt{n/2}\ .$$

Let us formulate the corresponding theorem.

Theorem 3.3. *The number of the Chebyshev's segments in the train of the nth Catalan number is less than* $\sqrt{n/2}$.

The actual number of Chebyshev's segments is often substantially less. The bounds of the segments converge rapidly as it approaches the power border, and near the power border many segments (and no-go zones also) are "collapse", i.e. is empty, without primes.

The bounds of segments are fractional numbers in most cases. It is more convenient to work with integer values, but rounding intervals should be kept open bounds. In the bottom of the segment we will truncate the fractional part (rounding down "a floor"), and for the top it is taken the nearest integer from above (rounding up "a ceiling").

Example 3.2. Consider the prime factorization of the millionth Catalan number (in the last section there are real programs for test calculations). The natural form of this number is not interesting, because the array of 600 thousand decimal digits has little to say. It's different with primes, even if a lot of them. Multiset $F_{1000000}$ includes 101543 primes and prime powers. All primes are selected from the factor base ${}_{\mathrm{p}}(1,\ 2{\cdot}10^6)$.

The base is separated by the border $\mathrm{ppb}(10^6) = \sqrt{2 \times 10^6} = 1414.21$ into two disjoint areas. In the upper base there are 707 Chebyshev's segments, which contain 99.7% of all prime factors. For example, in the segment $S_1 = {}_{\mathrm{p}}(1000001,\ 2{\cdot}10^6)$ there are 70435 primes, and the second $S_2 = {}_{\mathrm{p}}(5{\cdot}10^5,\ 666667)$ includes 12531 primes.

The nearest prime 1423 above the power border falls into the segment $S_{703} = {}_{\mathrm{p}}(1422,\ 1424)$. Consequently, four segments 704-707 are empty, i.e. don't contain primes. The next prime $1433 \mid C(10^6)$ select by the segment $S_{698} = {}_{\mathrm{p}}(1432,\ 1434)$. As a result, we have another group of empty segments numbered 699-702. As you can see, near the power border only the fifth segment "brings prey", i.e. it contains a prime that divides $C(10^6)$.

Example 3.3. For the 100,000,000th Catalan number ppb(10^8) = 14142, so the train primes are selected from the interval $_p(14142, 2{\cdot}10^8)$. Let's check two primes p =463,219 and q=543,061. To solve this problem it is enough to compare this numbers with the bounds of several Chebyshev's segments.

Choose a suitable segment for the first prime. Let's use the formula lower bounds: $(10^8+1)/463219 = 215.88$. The prime p is located below the segment 215, but above the lower limit of the segment 216. It remains to compare p with the upper limit of the segment 216. Let's check it: $\lceil 2{\cdot}10^8/(2{\cdot}216-1)\rceil = 464{,}038 > p$. So, $p \in S_{216}$.

Note that we could get an integer in the formula of the lower bound. This is a rare case, but possible, e.g., for C(99,592,084) we get the following (99592084+1)/463219 = 215, i.e. the prime 463219 is the lower bound of 215th segment. In this case the prime is rejected immediately, since $463219 \nmid C(99{,}592{,}084)$.

Similar calculations show that the second prime q is placed into the no-go zone between S_{184} and S_{185}. Consequently, $543061 \nmid C(100{,}000{,}000)$.

As described in Example 3.3, the check boils down to the following two steps: (a) calculation of the adjacent segment over a given prime p; (b) compare p with the upper limit of the found segment. Let's state an important theorem, which makes it easy to check any prime number.

Theorem 3.4. *The prime $p > \sqrt{2n}$ divides the nth Catalan number if and only if*

$$p < \lceil 2n/(2k-1)\rceil, \text{ where } k = \lceil (n+1)/p\rceil, \ kp \neq n+1.$$

The following corollary follows immediately from Theorem 3.4.

Corollary 3.1. *The nth Catalan number is not divisible by prime $p > \sqrt{2n}$ if $p \mid n+1$.*

Sieve of Chebyshev. To get a Catalan number train we can move the contents of the Chebyshev's segments in an empty set, e.g., starting from the main segment. But there is a faster way, which is similar to the algorithm of the sieve of Eratosthenes. Obviously, it is easier and faster to clear all no-go zones at the upper factor base, keeping the contents of the segments. The result is a construct that by analogy we will call the *Sieve of Chebyshev*.

4. The proofs of Theorem 3.1 and Theorem 3.2

Let us introduce some definitions and notations that will be needed in the future.

Power function is often found in the literature. For a prime p and a positive integer m, let $v_p(m)$ denotes the number of factor of p in the prime factorization m [Pom13]. For example, $v_7(14)=1$, $v_7(98)=2$, $v_7(5!)=0$. Here are a few properties of the power function:

$$v_p(ab) = v_p(a) + v_p(b), \ \ v_p(a^k) = k\,v_p(a), \ \ v_p(a/b) = v_p(a) - v_p(b).$$

Various properties of the power function see [Ep15]. This function is used in a prime factorization of special numbers. For example, the power of the odd prime p in the Cayley formula (2) is given by:

$$v_p(C(n)) = v_p((2n–1)!!) – v_p((n+1)!), \quad p > 2. \tag{6}$$

The last operand in (6) can be calculated using Legendre's formula [Pom13]:

$$v_p((n+1)!) = \sum_{j>0} \lfloor (n+1)/p^j \rfloor. \tag{7}$$

Odd rounding "floor". For a real $x \geq 0$, let $\lfloor x \rfloor_{\text{odd}}$ denote the rounding "floor" to the nearest odd integer, i.e. the fractional part of x is discarded with an additional decrease to 1 if the result is an even number or zero. Let's call this operation an *odd rounding "floor"*. For example, $\lfloor 23/7 \rfloor_{\text{odd}} = \lfloor 34/7 \rfloor_{\text{odd}} = 3$, $\lfloor 6/7 \rfloor_{\text{odd}} = –1$.

This operation is useful to determine the power of a prime in a double factorial. For example, the first operand in (6) can be calculated as follows (see [Ep15]):

$$v_p((2n–1)!!) = ½ \sum_{j>0} (\lfloor (2n–1)/p^j \rfloor_{\text{odd}} + 1). \tag{8}$$

The proof of Theorem 3.1. It is necessary to show that the kth segment of the nth Catalan number has the open top bound $u = 2n/(2k–1)$. When $k = 1$ we get the top base $2n$, since the main segment is adjacent to the open top of FB(n).

Let u be an integer and then u is an even number. If is not, and u is a fractional number, then round u to the nearest even number. Let $p = u–1$ and $q = u+1$ be primes. Obviously, $\text{ppb}(n) = (2n)^{½} < p < u < q$. So, we chose the most unfavorable conditions for p, u, q. Let us prove that $p \mid C(n)$ and $q \nmid C(n)$. Obviously,

$$n = ½u(2k–1) = ½(p+1)(2k–1) = ½(q–1)(2k–1).$$

Since $p^k > 2n$, $k>1$, then the equalities (7-8) have only one term in the sums, i.e.

$$v_p((n+1)!) = \lfloor (n+1)/p \rfloor \text{ and } v_p((2n–1)!!) = ½(\lfloor (2n–1)/p \rfloor_{\text{odd}} + 1), \; p > (2n)^{½}.$$

The same is true for q. To prove the theorem it is enough to confirm the equalities

$$v_p(C(n)) = 1, \; n = ½(p+1)(2k–1), \text{ and } v_q(C(n)) = 0, \; n = ½(q–1)(2k–1). \tag{9}$$

Let us first consider the ordinary factorials from (6).

$$v_p((n+1)!) = \lfloor (½(p+1)(2k–1)+1)/p \rfloor$$

$$= \lfloor k + k/p – ½ + 1/2p \rfloor = \begin{cases} k-1, & \text{if } k < p/2. \\ k, & \text{if } k \geq p/2. \end{cases} \tag{10}$$

$$v_q((n+1)!) = \lfloor (½(q–1)(2k–1)+1)/q \rfloor$$

$$= \lfloor k – k/q – ½ + 3/2q \rfloor = \begin{cases} k-1, & \text{if } k \leq q/2. \\ k-2, & \text{if } k > q/2. \end{cases} \tag{11}$$

The operands $^1\!/_{2p}$ and $^3\!/_{2q}$ are removed, since they do not affect the final result. In (10-11) only the top branches are real, so $v_p((n+1)!) = v_q((n+1)!) = k–1$.

Let's calculate the odd double factorials from (6).

$$v_p((2n–1)!!) = ½(\lfloor((p+1)(2k–1)–1)/p\rfloor_{\text{odd}} +1) =$$
$$= ½(\lfloor 2k–1 + {}^2\!/_p\,(k–1)\rfloor_{\text{odd}} +1) = ½((2k–1) +1) = k.$$

$$v_q((2n–1)!!) = ½(\lfloor((q–1)(2k–1)–1)/q\rfloor_{\text{odd}} +1) =$$
$$= ½(\lfloor 2k–1 – 2k/q\rfloor_{\text{odd}} +1) = ½((2k–3) +1) = k –1.$$

Recent calculations confirm (9). The proof of the Theorem 3.1 is complete.

The proof of Theorem 3.2. The open bottom of the Chebyshev's segment may be a prime number in contrast to the top bound. In this regard, we will show first that

$$p = (n+1)/k > \text{ppb}(n),\;\; k < ½\text{ppb}(n),$$

does not divide $C(n)$. In other words, by means of (6) let us prove the equality

$$v_p((2n–1)!!) = v_p((n+1)!),\;\; n = kp –1. \qquad (12)$$

Factorials (7-8) are free from squares of $p > (2n)^{½}$, therefore

$$v_p((n+1)!) = \lfloor(kp)/p\rfloor = k,$$
$$v_p((2n–1)!!) = ½(\lfloor(2kp–3)/p\rfloor_{\text{odd}} +1) = ½(\lfloor 2k–3/p\rfloor_{\text{odd}} +1) = ½(2k–1+1) = k.$$

So, $(n+1)/k \nmid C(n)$. Next, check the probably least prime of the kth segment. Let $w = (n+1)/k$ be an even integer, and let $q = 1+w < 2n/(2k–1)$ be a prime. We will show that $q \mid C(n)$. Obviously, $n = k(q–1)–1$, $q > w > \text{ppb}(n) > 2k$ or $q \geq 2k+3$. Let's compute factorials.

$$v_p((n+1)!) = \lfloor k(q–1)/q\rfloor = \lfloor k– {}^k\!/_q\rfloor = k–1.$$

$$v_p((2n–1)!!) = ½(\lfloor (2(k(q–1) –1)–1)/q\rfloor_{\text{odd}} +1) =$$
$$= ½(\lfloor 2k–(2k+3)/q\rfloor_{\text{odd}} +1) = ½(2k–1+1) = k.$$

Thus, in the kth Chebyshev's segment the probably least prime $1+(n+1)/k$ is guaranteed divides $C(n)$. The proof of the Theorem 3.2 is complete.

5. Slight Catalan numbers

The prime factors of a Catalan number that exceed the power border are chosen from the set of primes based on the segment bounds. Thus, the vast majority of primes are known, and Chebyshev segments cannot be stored. The train is ballast; it can be removed and, if necessary, we can easily restore. It is enough to keep the core of Catalan numbers. Below is selected information for the first Catalan numbers.

Index	Power border	Slight Catalan number	Prime factorization of Catalan number		Catalan number «heavyweight»
			core	train	
3	2,45	1		5	5
4	2,83	2	2	7	14
5	3,16	6	2, 3	7	42
6	3,46	12	2, 2, 3	11	132
7	3,74	3	3	11, 13	429
8	4,00	2	2	5, 11, 13	1430
9	4,24	2	2	11, 13, 17	4862
10	4,47	4	2, 2	13, 17, 19	16796
11	4,69	2	2	7, 13, 17, 19	58786
12	4,90	4	2, 2	7, 17, 19, 23	208012
13	5,10	100	2, 2, 5, 5	17, 19, 23	742900
14	5,29	360	2, 2, 2, 3, 3, 5	17, 19, 23	2674440
15	5,48	45	3, 3, 5	17,19, 23, 29	9694845
16	5,66	90	2, 3, 3, 5	19, 23, 29, 31	35357670
17	5,83	30	2, 3, 5	11, 19, 23, 29, 31	129644790
18	6,00	300	2, 2, 3, 5, 5	7, 11, 23, 29, 31	477638700
19	6,16	30	2, 3, 5	7, 11, 23, 29, 31, 37	1767263190
20	6,32	60	2, 2, 3, 5	7, 11, 13, 23, 29, 31, 37	6564120420

A slight Catalan number is significantly shorter than its ancestor "heavyweight" for two reasons: firstly, the core is very tiny in size and, secondly, the core contains the smallest factors. The difference is especially noticeable on large indices, for example, the 170th slight Catalan number is 4080, while in the corresponding progenitor has a length of 99 digits:

$C(170)$ = 566 348 408 726 522 751 148 449 775 858 720 556 691 622 190 005 859 389 137 334 807 741 187 245 136 563 759 042 704 144 561 379 440.

Of course, and slight numbers look impressive on the big indices. The slight 50,000th Catalan number has a length of 163 marks, here is this number:

1 029 142 440 334 210 758 480 708 051 574 203 810 960 548 889 204 183 685 792 756 307 454 455 534 384 071 475 346 148 063 228 602 598 951 202 112 317 923 378 920 856 767 137 362 573 866 245 850 058 506 779 182 608 960.

But the "heavyweight" with the index 50,000 is problematic to show because it "weighs" 30,000 digits. Among the slight Catalan numbers there are "doubles".

The small sizes of the slight numbers give the opportunity to learn the "depth" of the sequence of Catalan number. And the core is more interesting than a naturalized slight Catalan number itself. Let's show the 100 slight Catalan numbers (the first four items equal to 1):

00-09: 1, 1, 1, 1, 2, 6, 12, 3, 2, 2;
10-19: 4, 2, 4, 100, 360, 45, 90, 30, 300, 30;
20-29: 60, 60, 120, 450, 36, 1764, 392, 28, 280, 56;
30-39: 112, 7, 294, 1470, 84, 14, 28, 28, 1400, 490;
40-49: 980, 3780, 7560, 18900, 2520, 2520, 35280, 4410, 900, 36;
50-59: 216, 108, 216, 840, 336, 12, 24, 24, 240, 72;
60-69: 1008, 121968, 11616, 45375, 18150, 1650, 3300, 11550, 1039500, 29700;
70-79: 59400, 4950, 108900, 544500, 2134440, 1067220, 27720, 83160, 831600, 20790;
80-89: 1540, 10780, 21560, 84700, 33880, 5725720, 34354320, 780780, 273273000, 18218200;
90-99: 400400, 200200, 400400, 2002000, 8808800, 34684650, 69369300, 1415700, 5577000, 111540.

6. Software service

In conclusion, let us consider a software system for the implementation of test calculation in the real time (for more details see [Ep15]). Let's list some programs with brief description of their functions.

The small program can produce a list of primes in a certain range; the start of the range does not exceed 10^{10}. A prime factorization of the Catalan numbers can be performed in several ways. In the first variant, a Catalan number is displayed in the naturalized form (the maximum index is 10^4). In the second case, a natural number is not displayed (the index up to 10^7). There is a third option, in which only a slight Catalan number is decomposed (the index up to 10^8). A separate program allows you to get the core of the Catalan number (the index is practically unlimited).

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